ON SOME OPTIMUM ROCKET TRAJECTORIES

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PMM Vol.25, No.6, 1961, pp. 978-982

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(Received April 24, 1961)

In a recently published paper [1] two problems dealing with the optimization of rocket trajectories were treated by variational methods. These problems concern the determination of the optimum thrust program – magnitude and direction – resulting in minimum flight time or minimum fuel consumption for a rocket transferring between specified velocities in the horizontal and vertical planes, respectively.

The following assumptions were made in [1] and are retained here:

a) The rocket is a particle of variable mass, i.e. the moments of inertia are negligible;

b) The thrust direction is ideally controllable, i.e. can be changed instantaneously;

c) the acceleration of gravity is constant;

d) the aerodynamic forces are negligible;

e) thrust is proportional to mass flow rate [2].

Since the conclusions reached in [1] are not in agreement with those published earlier for one of the problems (Flight in Vertical Plane), another analysis is presented here. The method of solution employed here is based on the classical Calculus of Variations [3] with the inclusion of inequality constraints on control variables as described in [4].

Flight in Horizontal Plane

Equations of motion and statement of problem. For flight restricted to the horizontal plane Figs. 1 and 2 portray the geometry of the flight path and the force system, respectively. The flight path is described with respect to an inertial coordinate system $Oxyz$ with Oxy in the horizontal plane and Oz vertical upward.

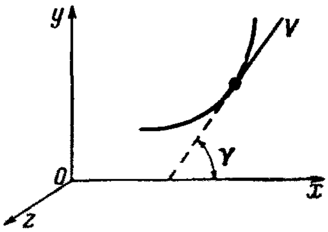

Fig. 1.

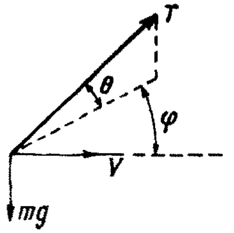

Fig. 2.

Upon decomposition of forces in the tangential, normal and binormal directions, respectively, one has

$$m\dot{V} = T \cos\theta \cos\varphi, \quad m\dot{V}\gamma = T \cos\theta \sin\varphi, \quad 0 = T \sin\theta - mg \tag{1}$$

where

$$T = c\beta, \quad c = \text{const}, \quad \beta = -\dot{m} \tag{2}$$

Since all possible control of the thrust direction can be effected with

$$0 \leqslant \varphi \leqslant 2\pi, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \tag{3}$$

one may write from $(1)_3$

$$\cos\theta = \left[1 - \left(\frac{mg}{T}\right)^2\right]^{1/2} \tag{4}$$

Furthermore, to insure a real solution one must impose the constraint

$$\frac{mg}{c} \leqslant \beta \leqslant \beta_{\max} \tag{5}$$

where $\beta_{\max}$ is prescribed.

It is required to minimize a functional

$$G = G(m_i, V_i, \gamma_i, t_i, m_f, V_f, \gamma_f, t_f) \tag{6}$$

subject to constraints

$$\begin{aligned} \dot{V} - [(c\beta / m)^2 - g^2]^{1/2} \cos\varphi &= 0 \\ \dot{\gamma} - [(c\beta / m)^2 - g^2]^{1/2} V^{-1} \sin\varphi &= 0 \\ \dot{m} + \beta &= 0 \\ (\beta_{\max} - \beta)(\beta - mg / c) - \eta^2 &= 0 \end{aligned} \tag{7}$$

and to appropriate end conditions. Equation $(7)_4$ arises from adjoining inequality constraint (5), where η is a real variable [5]. If either V_i or V_f is zero, a limiting process described in [6] must be employed.

Equations (7) contain the six dependent variables V, γ, m, β, ϕ and η. Since there are four constraint equations, two variables can be varied freely. These are the control variables β and ϕ.

First variation. Necessary conditions for the existence of an *extremal* value of G, arising from the vanishing of the first variation, are the Euler-Lagrange equations

$$\dot{\lambda}_V - \lambda_\gamma\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{1/2}\frac{1}{V^2}\sin\varphi = 0 \tag{8}$$

$$\dot{\lambda}_\gamma = 0$$

$$\dot{\lambda}_m - \frac{c^2\beta^2}{m^3}\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{-1/2}\left(\lambda_V\cos\varphi + \lambda_\gamma\frac{1}{V}\sin\varphi\right) + \lambda_\eta(\beta_{\max} - \beta)\frac{g}{c} = 0$$

$$\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{1/2}\left(\lambda_V\sin\varphi - \lambda_\gamma\frac{1}{V}\cos\varphi\right) = 0$$

$$\frac{c^2\beta}{m^2}\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{-1/2}\left(\lambda_V\cos\varphi + \lambda_\gamma\frac{1}{V}\sin\varphi\right) - \lambda_m + \lambda_\eta\left(2\beta - \beta_{\max} - \frac{mg}{c}\right) = 0$$

$$\lambda_\eta\eta = 0$$

and the transversality condition

$$dG + [\lambda_V\,dV + \lambda_\gamma\,d\gamma + \lambda_m\,dm - C\,dt]_{t_i}^{t_f} = 0 \tag{9}$$

where the first integral

$$C = \lambda_V\dot{V} + \lambda_\gamma\dot{\gamma} + \lambda_m\dot{m} = \text{const} \tag{10}$$

The λ_Z, $Z = V, \gamma, m, \eta$, are undetermined multipliers.

Corner conditions. Since the control variables β and ϕ may have finite discontinuities at isolated points of the interval $t_i \leqslant t \leqslant t_f$, the extremal arc may possess corners. At such corners the Weierstrass-Erdmann Corner Conditions apply. For the problem under consideration these are

$$\lambda_{Z-} = \lambda_{Z+}, \quad Z = V, \gamma, m \tag{11}$$

and

$$C_- = C_+ \tag{12}$$

Composition of the extremal arc. Since the control variables β and ϕ may be discontinuous, there arises the question of composing the subarcs into the total extremal arc.

From Equations $(7)_4$ and $(8)_6$ it follows that when

$$\frac{mg}{c} < \beta < \beta_{max} \quad \textbf{for} \quad \lambda_\eta = 0, \quad \eta \neq 0$$

$$\beta = \frac{mg}{c} \quad \text{or} \quad \beta = \beta_{max} \quad \textbf{for} \quad \begin{cases} \lambda_\eta = 0, & \eta = 0 \\ \lambda_\eta \neq 0, & \eta = 0 \end{cases} \tag{13}$$

bearing out the contention that the thrust magnitude is bounded. Unfortunately, conditions (13) yield no answer to the question of choosing the optimum thrust regime.

A similar dilemma arises with respect to the optimum choice of thrust direction angle ϕ, since Equation $(8)_4$ is satisfied by

$$mg = c\beta \quad \text{or} \quad \begin{cases} \sin\varphi = \pm\lambda_\gamma\,[\lambda_\gamma^2 + (V\lambda_V)^2]^{-1/2} \\ \cos\varphi = \pm V\lambda_V\,[\lambda_\gamma^2 + (V\lambda_V)^2]^{-1/2} \end{cases} \tag{14}$$

To resolve these questions one must turn to a stronger condition than that arising from the first weak variation.

Weierstrass *E*-function. For the minimum flight time problem

$$G = t_f \tag{15}$$

with

$$\begin{aligned} t = t_i = 0, \quad & V = V_i, \quad \gamma = \gamma_i, \quad m = m_i \\ t = t_f \text{ (unspecified)}, \quad & V = V_f, \quad \gamma = \gamma_f, \quad m = m_f \end{aligned} \tag{16}$$

so that Equations (9) and (10) yield

$$C = 1 \tag{17}$$

For the minimum fuel problem, or any other, with unspecified flight time

$$C = 0 \tag{18}$$

Consequently, from Equations (10) and (7) one has then

$$C = \left[\left(\frac{c\dot{\beta}}{m}\right)^2 - g^2\right]^{1/2} \left(\lambda_V \cos\varphi + \lambda_\gamma \frac{1}{V}\sin\varphi\right) - \lambda_m\beta \geqslant 0 \tag{19}$$

Along a programmed intermediate thrust arc conditions (13) require

$$\lambda_\eta = 0 \tag{20}$$

whence Equation $(8)_5$ becomes

$$C = -g^2\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{1/2} \left(\lambda_V \cos\varphi + \lambda_\gamma \frac{1}{V}\sin\varphi\right) \tag{21}$$

In order that functional G have a minimum the Weierstrass E-Function must be non-negative. This condition requires that

$$A \equiv \left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{1/2} \left(\lambda_V \cos\varphi + \lambda_\gamma \frac{1}{V}\sin\varphi\right) - \lambda_m\beta \tag{22}$$

be maximum with respect to the control variables β and ϕ.

With respect to thrust direction angle ϕ this requirement results in Equation (8.4) and the choice of the + sign in Equations (14), i.e.

$$\lambda_V \cos\varphi + \lambda_\gamma \frac{1}{V} \sin\varphi = [\lambda_\gamma^2 + (V\lambda_V)^2]^{1/2} \frac{1}{V} \geqslant 0 \tag{23}$$

In effect, only the inequality sign applies in inequality (23) lest all multipliers vanish. Consequently, Equation (21) contradicts inequality (19). Thus, there can be no arc of intermediate thrust, i.e.

$$\lambda_\eta \neq 0, \qquad \eta = 0 \text{ and } \beta = \frac{mg}{c} \quad \text{or} \quad \beta = \beta_{\max} \tag{24}$$

With respect to the bounded control variable β

$$\frac{\partial A}{\partial \beta} = \frac{c^2\beta}{m^2}\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{-1/2} [\lambda_\gamma^2 + (V\lambda_V)^2]^{1/2} \frac{1}{V} - \lambda_m \tag{25}$$

and

$$\frac{\partial^2 A}{\partial \beta^2} = -\left(\frac{cg}{m}\right)^2 [\lambda_\gamma^2 + (V\lambda_V)^2]^{1/2} \left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{-3/2} \frac{1}{V} \tag{26}$$

In view of Equations $(8)_5$, (24), and (25)

$$\partial A / \partial \beta \neq 0 \tag{27}$$

Also from Equation (25)

$$\partial A / \partial \beta \to \infty \qquad \text{as } c\beta \to mg \tag{28}$$

and from Equation (26)

$$\partial^2 A / \partial \beta^2 < 0 \tag{29}$$

Figure 3 then shows A as a function of β at any instant of time.

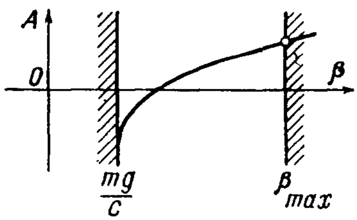

Fig. 3.

It follows from Fig. 3 that only maximum thrust is permissible, i.e.

$$\beta = \beta_{\max} \tag{30}$$

Furthermore, at that point

$$A = C \tag{31}$$

If

$$\beta_{\max} = \infty \tag{32}$$

which is the case treated in [1], the final velocity is obtained by means of an impulse.

If the flight time is specified, Equations (17) and (18) no longer hold. In particular, the first integral need no longer be non-negative and programmed intermediate thrust may be called for. Since A is not a linear function of β, the optimum thrust program must vary continuously to satisfy the maximality of A. When the maximum value of A, Equation (22), lies in the interval $mg/c < \beta < \beta_{max}$, the optimum program is given by

$$\partial A / \partial \beta = 0 \tag{33}$$

that is, Equation $(8)_5$ with Equation (20). When the value of A corresponding to Equation (33) lies to the right of $\beta = \beta_{max}$ in Fig. 3, then

$$\lambda_n \neq 0, \ \beta = \beta_{max} \tag{34}$$

where λ_η is given by Equation $(8)_5$.

Solution. For finite thrust the equations of motion and the variational equations must be solved together to provide the optimum thrust direction program.

For minimum flight time and end conditions (16) one may proceed by assuming initial values for λ_γ and λ_V. Equations $(7)_{1-3}$ and $(8)_{1-2}$ together with Equations (14) and (30) are then integrated until $m = m_f$. At that point the end values of V and γ must be matched. If the final mass m_f is not specified, the integration must be terminated when

$$\lambda_m = \lambda_{mf} = 0 \tag{35}$$

The multiplier λ_m may be computed from the first integral, Equations (10) and (17).

For minimum fuel consumption

$$G = -m_f \tag{36}$$

and

$$\begin{aligned} &t = t_i = 0, \quad V = V_i, \quad \gamma = \gamma_i, \quad m = m_i \\ &t = t_f \text{ (unspecified) } V = V_f, \quad \gamma = \gamma_f \end{aligned} \tag{37}$$

it follows from Equation (9) that

$$\lambda_{mf} = 1 \tag{38}$$

The integration is terminated when λ_m reaches unity. Again, λ_m may be found from the first integral, Equations (10) and (18).

In either case a two-parameter iteration is required for the solution of the mixed end value problem.

Flight in Vertical Plane

The problem of optimal flight in the vertical plane has been solved earlier [7]. Here again it can be shown that flight at maximum thrust is optimum for the problems considered.

Note. The results obtained in this paper do not agree with those of [1], where the solution calls for flight with programmed intermediate thrust.

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(Author's original)